Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021–8928(02)00129-6
J. Appl. Maths Mechs, Vol. 66, No. 6, pp. 873–879, 2002

0021–8928/02/\$—see front matter

INTEGRAL VARIATIONAL PRINCIPLES IN POINCARÉ AND CHETAYEV VARIABLES†

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(*Received* 23 *May* 2002)

A holonomic mechanical system with k degrees of freedom is considered, its state being characterized by $n \geqslant k$ defining coordinates, $p < k$ Poincaré parameters [1] and $k-p$ Chetayev parameters [2]. In these variables, generalized Routh equations are introduced and expressions are given for the integral variational principles of Hamilton–Ostrogradskii and Hamilton (the third form), as well as Hölder's principle and the Lagrange and Jacobi versions of the principle of least action.

1. THE ROUTH FUNCTION

Suppose a mechanical system is constrained by smooth holonomic constraints and its position in space at a time t is defined by real not necessarily independent variables x_i ($i = 1, \dots, n$). We define a virtual displacement of the system by a certain intransitive k-member group of infinitesimal operators [1, 2]

$$X_\alpha = \xi^i_\alpha \partial / \partial x_i, \quad \alpha = 1, \dots, k$$

Throughout this paper repeated indices represent summation.

The group of virtual displacements is defined by its structural coefficients $c^i_{\alpha\beta}$

$$[X_\alpha, X_\beta] \equiv X_\alpha X_\beta - X_\beta X_\alpha = c^i_{\alpha\beta} X_i, \quad \alpha, \beta, i = 1, \dots, k \tag{1.1}$$

The variations of a continuous differentiable function $f(t, x_1, \dots, x_n)$ under virtual and actual displacements are defined by the relations

$$\delta f = \omega_\alpha X_\alpha f, \quad df = (\partial f / \partial t + \eta_\alpha X_\alpha f) dt \tag{1.2}$$

respectively, it being assumed that the operator $\partial / \partial t$ commutes with the group of virtual displacements X_α

$$[\partial / \partial t, X_\alpha] = 0, \quad \alpha = 1, \dots, k$$

The quantities ω_α in (1.2) are independent virtual displacement parameters, and η_α are independent actual displacement parameters, introduced by Poincaré [1] and satisfying the relations

$$\delta \eta_i = d\omega_i / dt - c^i_{\alpha\beta} \omega_\alpha \eta_\beta, \quad \alpha, \beta, i = 1, \dots, k \tag{1.3}$$

in the case when the operators d and δ commute, $d\delta = \delta d$.

We shall assume that the kinetic energy of the system is a positive-definite quadratic form

$$T(x, \eta) = \tfrac{1}{2} a_{ij}(x) \eta_i \eta_j, \quad i, j = 1, \dots, k \tag{1.4}$$

Instead of part of the Poincaré parameters η_s, we will introduce new variables – the Chetayev parameters [2]

$$y_s = \partial T / \partial \eta_s = a_{si} \eta_i, \quad s = p+1, \dots, k, \quad 0 < p < k \tag{1.5}$$

and, using these equalities, express the parameters η_s as follows [3]

†*Prikl. Mat. Mekh.* Vol. 66, No. 6, pp. 914–921, 2002.

$$\eta_s = b_{sr} y_r - \gamma_{sj}\eta_j, \quad r,s = p+1,\ldots,k, \quad j = 1,\ldots,p \tag{1.6}$$

where $b_{rs} = A_{rs}/D, D = \det(\partial^2 T/\partial\eta_r\partial\eta_s)^k_{s,r=p+1} \neq 0, \gamma_{sj} = b_{sr}a_{rj}$ and A_{rs} denotes the cofactor of an element a_{rs} of the determinant D. Substituting Eqs (1.6) into (1.4) we obtain the following expression for the kinetic energy of the system in terms of Poincaré and Chetayev variables

$$T^*(x,\eta,y) = \tfrac{1}{2} a^*_{ij}\eta_i\eta_j + \tfrac{1}{2} b_{rs} y_r y_s \tag{1.7}$$

where

$$a^*_{ij} = a_{ij} - b_{rs}a_{rj}a_{si}, \quad i,j = 1,\ldots,p; \quad r,s = p+1,\ldots,k$$

Let us assume that the system is subject to both potential forces, with force function $U(t, x_1, \ldots, x_n)$, and non-potential generalized forces $Q_\alpha = \mathbf{F}_\nu \cdot X_\alpha \mathbf{r}_\nu$, where $\mathbf{F}_\nu$ is the non-potential force applied to a point mass with radius vector $\mathbf{r}_\nu$ ($\nu = 1, \ldots, N$). The Lagrangian will be denoted by

$$L(t,x,\eta) = T(x,\eta) + U(t,x)$$

We define the Routh function by

$$R(t,x,\eta,y) = L(t,x,\eta) - \eta_s y_s, \quad s = p+1,\ldots,k \tag{1.8}$$

where all the quantities η_s on the right are expressed in terms of ν_s by formulae (1.6), so that

$$R(t,x,\eta,y) = \tfrac{1}{2} a^*_{ij}\eta_i\eta_j + \gamma_{sj}\eta_j y_s - \tfrac{1}{2} b_{rs} y_r y_s + U(t,x_i) \tag{1.9}$$

Comparing the variations of both sides of (1.8), we obtain the relations

$$X_\alpha R = X_\alpha L, \quad \frac{\partial R}{\partial \eta_r} = \frac{\partial L}{\partial \eta_r}, \quad \frac{\partial R}{\partial y_s} = -\eta_s, \quad \alpha = 1,\ldots,k, \; r = 1,\ldots,p, \; s = p+1,\ldots,k \tag{1.10}$$

which hold because of the independence of the initial values of the coordinates and the velocities at the initial time, for which we are free to choose that considered [4].

2. DERIVATION OF THE INTEGRAL VARIATIONAL PRINCIPLES OF MECHANICS. SYNCHRONOUS VARIATION

Bearing equality (1.8) in mind and noting that the work done by the non-potential forces in the virtual displacements is $A = \omega_\alpha Q_\alpha$, we conclude that the Hamilton–Ostrogradskii principle [3] takes the following form in terms of Poincaré and Chetayev variables

$$\int_{t_0}^{t_1} [\delta(R + \eta_s y_s) + \omega_\alpha Q_\alpha] dt = 0; \quad \omega_\alpha = 0 \quad \text{for} \quad t = t_0, t_1; \; \alpha = 1,\ldots,k \tag{2.1}$$

This principle is a necessary and sufficient condition for actual motion of the mechanical system under the action of the applied forces. The actual motion (the "direct route") is assumed to be compared with the varied motions (the "indirect routes"), the configurations of the system being the same for all motions, at both the initial time t_0 and the final time t_1, which are chosen arbitrarily. Condition (2.1) leads to the equation

$$\int_{t_0}^{t_1} \left[\omega_i(X_i R + Q_i) + \frac{\partial R}{\partial \eta_r}\delta\eta_r + \left(\frac{\partial R}{\partial y_s} + \eta_s\right)\delta y_s + y_s \delta\eta_s\right] dt = 0$$

which, in view of relations (1.3), may be reduced via integration by parts to the form

$$\left(\frac{\partial R}{\partial \eta_r}\omega_r + y_s\omega_s\right)\Bigg|_{t_0}^{t_1} + \int_{t_0}^{t_1} \left[\left(-\frac{d}{dt}\frac{\partial R}{\partial \eta_r} - c^\beta_{r\alpha}\eta_\alpha \frac{\partial R}{\partial \eta_\beta} + X_r R + Q_r\right)\omega_r + \right.$$

$$\left. + \left(-\frac{dy_s}{dt} - c^\beta_{s\alpha}\eta_\alpha \frac{\partial R}{\partial \eta_\beta} + X_s R + Q_s\right)\omega_s + \left(\frac{\partial R}{\partial y_s} + \eta_s\right)\delta y_s\right] dt = 0; \quad \omega_i = 0 \quad \text{for} \quad t = t_0, \; t_1$$

The expression outside the integral vanishes by virtue of the conditions at the ends. Assuming that the quantities ω_r, ω_s and δy_s are arbitrary and independent in the interval (t_0, t_1) and taking expressions (1.5) into consideration, we obtain the generalized Routh equations of motion in Poincaré and Chetayev variables

$$\frac{d}{dt}\frac{\partial R}{\partial \eta_r} = \frac{\partial R}{\partial \eta_\beta}\left(c_{\alpha r}^{\beta}\eta_\alpha - c_{sr}^{\beta}\frac{\partial R}{\partial y_s}\right) + y_\delta\left(c_{\alpha r}^{\delta}\eta_\alpha - c_{sr}^{\delta}\frac{\partial R}{\partial y_s}\right) + X_r R + Q_r$$

$$\frac{dy_s}{dt} = \frac{\partial R}{\partial \eta_\beta}\left(c_{\alpha s}^{\beta}\eta_\alpha - c_{\gamma s}^{\beta}\frac{\partial R}{\partial y_\gamma}\right) + y_\delta\left(c_{\alpha s}^{\delta}\eta_\alpha - c_{\gamma s}^{\delta}\frac{\partial R}{\partial y_\gamma}\right) + X_s R + Q_s, \quad \eta_s = -\frac{\partial R}{\partial y_s} \tag{2.2}$$

$$r, \alpha, \beta = 1, \ldots, p, \quad s, \gamma, \delta = p+1, \ldots, k$$

If $p = k$ ($p = 0$), these equations become the Poincaré (Chetayev) equations.

In the case when the Routh function is not explicitly dependent on the time t and all the non-potential forces vanish, $Q_i = 0$ ($i = 1, \ldots, k$), Eqs (2.2) have an energy integral [3]

$$\eta_r\frac{\partial R}{\partial \eta_r} - R = T^* - U = h = \text{const}, \quad r = 1, \ldots, p \tag{2.3}$$

Chetayev [2] introduced the important concept of cyclic displacements in the case that there are no non-potential forces. Displacements X_α ($\alpha = p+1, \ldots, k$) are known as cyclic displacements if they satisfy the conditions

$$1)\ X_\alpha L = 0, \quad 2)\ \lfloor X_\alpha X_\beta \rfloor = 0, \quad \beta = 1, \ldots, k$$

By conditions 2, all the structural coefficients vanish, $c_{\alpha\beta}^{i} = 0$, if the subscript α or β equals one of the numbers $p+1, \ldots, k$. In that case, by relations (1.10), the second group of Eqs (2.2) yields first integrals

$$y_s = c_s = \text{const}, \quad s = p+1, \ldots, k \tag{2.4}$$

and the first group of Eqs (2.2) becomes [2]

$$\frac{d}{dt}\frac{\partial R}{\partial \eta_r} = c_{\alpha r}^{\beta}\eta_\alpha\frac{\partial R}{\partial \eta_\beta} + c_{\alpha r}^{\delta}\eta_\alpha c_\delta + X_r R + Q_r, \alpha, \ \beta, \ r = 1, \ldots, p, \ \ \delta = p+1, \ldots, k \tag{2.5}$$

After integration of these equations for the non-cyclic displacements, the Poincaré parameters η_s ($s = p+1, \ldots, k$) are defined by the relations

$$\eta_s = -\frac{\partial R}{\partial c_s}, \quad R = R(t, x_i, \eta_r, c_s)$$

When these are no non-potential forces, i.e. when $Q_i = 0$ ($i = 1, \ldots, k$), relations (2.1) imply the third form of Hamilton's principle in Poincaré and Chetayev variables

$$\delta\int_{t_0}^{t_1}(R + \eta_s y_s)dt = 0; \quad \omega_i = 0 \ \text{ for } \ t = t_0, t_1; \ \ s = p+1, \ldots, k, \ \ i = 1, \ldots, k \tag{2.6}$$

unlike the first form in Poincaré variables and the second form in Chetayev variables

$$\delta\int_{t_0}^{t_1} L(t, x, \eta)dt = 0 \ \text{ and } \ \delta\int_{t_0}^{t_1}[y_s\eta_s - H(t, x, y)]dt = 0; \quad \omega_i = 0 \ \text{ for } \ t = t_0, t_1 \tag{2.7}$$

where the Hamiltonian is

$$H(t, x, y) = y_s\eta_s - L(t, x, \eta), \quad s = 1, \ldots, k$$

The third form (2.6) of Hamilton's principle occupies an intermediate position between the first and second forms (2.7) of the principle. This form of the principle is significant in its own right, because of the assumption that the variations δy_s are arbitrary and independent of ω_i and $\delta\eta_r$ inside the interval (t_0, t_1). One should also bear in mind that in the general case the comparison curves may not satisfy the relations $y_s = \partial L/\partial\eta$.

3. DERIVATION OF THE INTEGRAL VARIATIONAL PRINCIPLES OF MECHANICS. ASYNCHRONOUS VARIATION

In the variational principles (2.1), (2.6) and (2.7) described above, we considered synchronous variations: a point P on the direct path at time t was associated with a point P' on the indirect path at the same instant of time. We shall now consider asynchronous variation, when the point P_i with coordinate x_i on an actual trajectory at time t is associated with the point P' with coordinates $x_i + \delta x_i$ on the varied trajectory at the time $t + \delta t$. The variations δx_i and δt are assumed to be functions of class C_2 and the relations between the Cartesian and defining coordinates of the system are independent of time. In asynchronous variation the operators d and δ do not commute, and instead of the equalities $d\delta = \delta d$ and (1.3) we have the formulae

$$\delta\frac{dx_i}{dt} = \frac{d}{dt}\delta x_i - \dot{x}_i\frac{d}{dt}\delta t,\quad \delta\eta_i = \frac{d\omega_i}{dt} - c^i_{\alpha\beta}\omega_\alpha\eta_\beta - \eta_i\frac{d}{dt}\delta t, \alpha,\beta,\quad i = 1,\dots,k \tag{3.1}$$

Using these relations and integration by parts, from the general equation of dynamics [4] we obtain, instead of (2.1), Hölder's principle [5]

$$\int_{t_0}^{t_1}\left[\delta(R+\eta_s y_s) + \left(\eta_r\frac{\partial R}{\partial\eta_r} + \eta_s y_s\right)\frac{d}{dt}\delta t - \frac{\partial R}{\partial t}\delta t + \omega_i Q_i\right]dt = 0;\quad \omega_i = 0\quad \text{for}\quad t = t_0, t_1 \tag{3.2}$$

if the parameters ω_i $(i = 1, \dots, k)$ are virtual displacements at each instant of time, vanishing at times t_0 and t_1, whereas the function δt does not necessarily vanish at $t = t_0, t_1$.

It is easily seen that

$$\eta_r\frac{\partial R}{\partial\eta_r} + \eta_s y_s = 2T^*,\quad r = 1,\dots,p,\quad s = p+1,\dots,k \tag{3.3}$$

by means of which (3.2) can be represented by the relation

$$\int_{t_0}^{t_1}\left[\delta(R+\eta_s y_s) + 2T^*\frac{d}{dt}\delta t - \frac{\partial R}{\partial t}\delta t + \omega_i Q_i\right]dt = 0;\quad \omega_i = 0\quad \text{for}\quad t = t_0, t_1 \tag{3.4}$$

which, like (3.2), expresses Hölder's principle in terms of Poincaré and Chetayev variables: if one compares the actual motion of a material system with a slightly different motion, in which the initial and final states remain unvaried and the displacements from each state of the actual motion to the corresponding state of the varied motion are virtual, then relations (3.2) and (3.4) hold.

Here the variation may be made even more specific by using the first or second mode of variation prescribed by Hölder [5].

1. In the case of synchronous variations, when the corresponding positions of the actual and varied motions are traversed simultaneously, that is, $\delta t \equiv 0$, Eqs (3.2) and (3.4) take the form of the Hamilton–Ostrogradskii principle (2.1) and, if also $Q_i = 0$ $(i = 1, \dots, k)$, Hamilton's principle (2.6).

2. In the case of asynchronous variations and no force functions, that is, when $U(t,x) = 0$, it follows from (1.7) and (1.8) that

$$R(x,\eta,y) + \eta_s y_s = T^*(x,\eta,y),\quad \frac{\partial R}{\partial t} = 0 \tag{3.5}$$

Set

$$\delta T^* = \omega_i Q_i,\quad i = 1,\dots,k \tag{3.6}$$

that is, we are requiring the difference between the kinetic energies for the corresponding states of both motions to equal the work that would be performed by the actual forces in the displacements relating corresponding positions. This determines in what way the system goes through the continuous sequence of varied positions. For this special mode of variations, Eqs (3.4)–(3.6) lead to the relation

$$\int_{t_0}^{t_1} 2(\delta T^* dt + T^* d\delta t) = \delta \int_{t_0}^{t_1} 2T^* dt = 0; \ \omega_i = 0 \ \text{ for } \ t = t_0, t_1 \tag{3.7}$$

This is the principle of least action, derived by Hölder, in its extended form. The more restricted form of the principle of least action will be worked out in the next two sections.

4. LAGRANGE'S VERSION OF THE PRINCIPLE OF LEAST ACTION

Let us assume that the Routh function is not explicitly dependent on time and that the non-potential forces vanish, $Q_i = 0$, so that energy integral (2.3) exists. The mechanical system, left to itself, may choose its motions from motions with a given total energy h, which makes it possible to limit the set of comparable trajectories by condition (2.3) [4].

Under the conditions $\partial R/\partial t = 0$, $Q_i = 0$ $(i = 1, \dots, k)$, Hölder's principle (3.4) becomes

$$\int_{t_0}^{t_1} \left[\delta(R + \eta_s y_s) + 2T^* \frac{d}{dt}\delta t\right] dt = 0; \quad \omega_i = 0 \ \text{ for } \ t = t_0, t_1; \quad s = p+1, \dots, k \tag{4.1}$$

Using the relation

$$R = \eta_r \partial R / \partial \eta_r - h \tag{4.2}$$

which follows from energy integral (2.3), we can express Eq. (4.1) in the form

$$\int_{t_0}^{t_1} \left(2\delta T^* + 2T^* \frac{d}{dt}\delta t\right) dt - (t_1 - t_0)\delta h = 0$$

which implies the Lagrange version of the principle of least action in Poincaré and Chetayev variables

$$\delta \int_{t_0}^{t_1} 2T^* dt = 0, \quad \delta h = 0; \quad \omega_i = 0 \quad \text{for} \quad t = t_0, t_1; \quad i = 1, \dots, k \tag{4.3}$$

The actual motion of a conservative holonomic system between two given configurations differs from the kinematically possible motions which take place between the same two configurations, with the energy h as the actual motion, in that the total variation of the Lagrange action $\int_{t_0}^{t_1} 2T^* \, dt$ for the actual motion has a stationary value.

We note that, thanks to the existence of the energy integral, the time taken by the system to go from one position to another depends on the path and is determined by it, so that the upper limit t_1 of the integral in (4.3) will be variable, and the variation of the integral (4.3) must be total.

Obviously, the principle (4.3) is analogous to its form in terms of Poincaré variables ($T^* = T$).

The principle of least action, like Hamilton's principle, expresses a necessary and sufficient condition for actual motion, and the equations of motion may be derived from it. In fact, let us consider the Lagrangian with multiplier λ for the conditional variational problem (4.3)

$$F = 2T^* + \lambda(T^* - U - h)$$

The transversality condition for the variable end at the upper limit t_1 of the integral

$$F - \eta_r \frac{\partial F}{\partial \eta_r} - \eta_s \frac{\partial F}{\partial \eta_s} = 0, \quad r = 1, \dots, p; \quad s = p+1, \dots, k$$

yields the equality $-2(1+\lambda)T^* = 0$, $\lambda = -1$, and, taking (1.8) into account, we obtain

$$F = R + \eta_s y_s + h$$

Consequently, Euler's equations for the variational problem with integrand F, in terms of Poincaré and Chetayev variables, have the form of the equations of motion (2.2) for $Q_i = 0$ $(i = 1, \dots, k)$.

Note that if (4.2) is used to replace the term $\eta_r \partial R/\partial \eta_r$ in Hölder's principle (3.2) by $R + h$, then, conditional upon $\partial R/\partial t = 0$, $Q_i = 0$, we obtain

$$\int_{t_0}^{t_1} \left[\delta(R + \eta_s y_s) + (R + \eta_s y_s + h)\frac{d}{dt}\delta t \right] dt = 0; \quad \omega_i = 0 \quad \text{for} \quad t = t_0, t_1$$

that is,

$$\delta \int_{t_0}^{t_1} (R + \eta_s y_s) dt = -h\delta t \Big|_{t_0}^{t_1}; \quad \omega_i = 0 \quad \text{for} \quad t = t_0, t_1$$

This yields the generalized Hamilton principle [6] in Poincaré and Chetayev variables

$$\delta \int_{t_0}^{t_1} (R + \eta_s y_s + h) dt = 0; \quad \omega_i = 0 \quad \text{for} \quad t = t_0, t_1 \tag{4.4}$$

5. JACOBI'S VERSION OF THE PRINCIPLE OF LEAST ACTION

Jacobi [7] eliminated the time t from Lagrange's principle by using the energy integral, and reduced it to space elements, thereby giving the principle of least action a geometrical aspect [4]. In addition, he showed that the integral does not reach a minimum between any two positions of the system, but only when the initial and final position are sufficiently close together.

Let us choose some new independent variable τ in such a way that its values lie between time-independent limits τ_0 and τ_1. As τ one might take, e.g., one of the coordinates x_i which varies monotonically with t in the interval under consideration [6, 7]. When the system is in motion, the coordinates x_i $(i = 1, \dots, n)$ will be certain functions of the variables τ; we denote their derivatives with respect to that variable by $x_i' = dx_i/d\tau$. We denote the Poincaré parameters corresponding to the velocities x_i' by $\tilde{\eta}_s$ and, as in the treatment of (1.4), we consider the quadratic form

$$\tilde{T}(x, \tilde{\eta}) = \tfrac{1}{2} a_{ij} \tilde{\eta}_i \tilde{\eta}_j$$

We then have

$$T = \tilde{T}\left(\frac{d\tau}{dt}\right)^2 = U + h, \quad \frac{d\tau}{dt} = \sqrt{\frac{U+h}{\tilde{T}}} \tag{5.1}$$

Bearing in mind that $T = T^*$, we write the principle (4.3) in Poincaré variables x_i, $\tilde{\eta}_s$ in the form

$$\delta \int_{\tau_0}^{\tau_1} 2\sqrt{\tilde{T}(U+h)}\, d\tau = 0; \quad \delta h = 0; \quad \omega_i = 0 \quad \text{for} \quad \tau = \tau_0, \tau_1; \quad i = 1, \dots, k \tag{5.2}$$

This equality is Jacobi version of the principle of least action.

In actual motion, the Jacobi's action takes a stationary value compared with its values for infinitely close adjacent motions that take the system from the same initial position to the same final position, with the first of Eqs (5.1) observed and the same value of the constant h as in the actual motion.

Thus, the Jacobi principle reduces the problem of determining the trajectory of the representative point in x-space to a problem of the variational calculus (5.2) with fixed endpoints. The velocity of motion of the representative point along the trajectory is found from the energy integral.

In conclusion, let us express the Jacobi principle in terms of Poincaré and Chetayev variables. Comparing the principle (5.2) with the first form of Hamilton's principle (2.7), we see that the integrand in (5.2) may be taken as a new Lagrangian $\tilde{L}(x_i, \tilde{\eta}_j)$ with independent variable τ and velocities x_i' [7]. By analogy with the function (1.8), we introduce the Routh function

$$\tilde{R}(x_i, \tilde{\eta}_r, \tilde{y}_s) = \tilde{L}(x_i, \tilde{\eta}_j) - \tilde{\eta}_s \tilde{y}_s, \quad \tilde{L}(x_i, \tilde{\eta}_j) = 2\sqrt{\tilde{T}(U+h)} \tag{5.3}$$

where

$$\tilde{y}_s = \frac{\partial \tilde{L}}{\partial \tilde{\eta}_s} = \frac{\partial \tilde{T}}{\partial \tilde{\eta}_s}\sqrt{\frac{U+h}{\tilde{T}}} = y_s, \quad s = p+1, \dots, k$$

Comparing the variations of both sides of Eq. (5.3), we find relations similar to (1.10). As a result we obtain an expression for the Jacobi principle in Poincaré and Chetayev variables

$$\delta \int_{\tau_0}^{\tau_1} [\tilde{R}(x_i, \tilde{\eta}_r, y_s) + \tilde{\eta}_s y_s] d\tau = 0; \quad \delta h = 0; \quad \omega_i = 0 \quad \text{for} \quad \tau = \tau_0, \tau_1 \tag{5.4}$$

It is readily seen that the equations of the extremals of the variational problem (5.4), when (5.1) is used to return to the independent variable t, take the form of the equations of motion (2.2) with $Q_i = 0$ $(i = 1, \dots, n)$.

This research was supported financially by the Russian Foundation for Basic Research (02-01-00196) and the Federal Special-Purpose "Integration" Programme.

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Translated by D.L.